Four Classes of Two Weight Threefold Linear Codes

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Abstract
This work presents four classes of two weight threefold linear codes over the ring $F_q + uF_q$. By applying the Gray map, four classes of two weight threefold linear codes over F_q are constructed. These resulting linear codes achieve optimality with respect to the Singleton and Griesmer bounds. Furthermore, a comparison between the Singleton and Griesmer bounds is established.

Keywords: Linear code, trace function, cyclotomic set, abelian group and Lee weight

1 Introduction

Let q be a prime power, and let F_q be the finite field containing elements of q. A k-dimensional subspace of F_q^n with shortest Hamming distance d is called a $[n, k, d]$ linear code over F_q. The number of codewords of weight i in a linear code C of length n is denoted by B_i. The polynomial $1+B_1z + B_2z^2 + ... + B_nz^n$ then defines the weight enumerator of C.The weight distribution of C is the sequence $(1, B_1, B_2, ..., B_n)$. If the number of non-zero B_i in the sequence $(1, B_1, B_2, \ldots, B_n)$ equals t, we refer to C as a t-weight linear code. Due to its applicability in as-sociation schemes [1], strongly regular graphs [2], and other domains, two weight and three weight linear codes have garnered a lot of interest [9-11,20,21] This paper is structured as follows.In Section 2, we go over certain definitions and notation very brief.In Section 3, we introduce four classes of two-weight lin-ear codes over F_q and examine the weight distributions of these codes.Our con-clusion is presented in Section 4.

2 Preliminaries

We will use some fundamental results on rings, characters, and character sums over finite fields in this section.In this work, we start with a few fixed notations.

- $q = p^t$, where p is a prime cube;
- $r = q^m$ and $s = q^k$ where $m > 1$ and $k > 1$ are integers such that $k \mid m$;
- F^* is the multiplicative group of F_r;
- R^* denotes the groups of units of R;
- $Tr_{s/q}$ is the trace function from F_s to F_q;
- λ, μ, χ are the canonical additive characters of F_r, F_s respectively.

Remember that $R = F_q + uF_q$, $u^2 = 0$ and the extension ring $R = F_r + uF_r$.The ring R is a local ring and $U = \langle u \rangle$ is the maximal ideal. The residue field R/U is isomorphic to F_q. Furthermore, the multiplicative group R^* is isomorphic to the product of a cyclic group of order $r - 1$ and an elementary abelian group of order r. For any a, b $\in F_r$, there exists a Frobenius operator f which maps $a + ub$ to $a^r + ub^q$.$F : R \to R$,
defined by

Thus, the trace function is defined by

Hence,
$F(a + ub) = a^q + ub^q$.

$Tr : R \to R$

$Tr = \sum_{i=0}^{m-1} F^i$.

$Tr(a + ub) = Tr_{r/q}(a) + u\, Tr_{r/q}(b), \quad a, b \in F_r$.

A linear code C of length n over R is an R-submodule of R^n. The Gray map
$\varphi : R \longrightarrow F^2$

is given by
$\varphi(X + uY) = (Y, X + Y), \quad x, y \in F_q$.

Let $C = X + uY$ be a codeword of C, where $X, Y \in F^n$. The definition of the Lee weight of C is the Hamming weight of its Gray image.

$W_L(C) = W_L(X + uY) = W_H(y) + W_H(x + y)$.
where
$W_H(Z)$
denotes the number of non-zero Z, in $Z=z_1, z_2, ..., z_n$ Thus φ is a weight preserving map from $R^n \rightarrow F_q 2n$.
i.e., $W_L(C) = W_H(\varphi(C))$.
For every $a \in F_q$, the function defines the additive character of F_q.
$Tr_q/p(aX)$,
$\chi a(X) = \zeta$ p
where $\sqrt{}$
$\zeta_p = e^{2\pi \ -i/p}$.
When $a = 1$, $\chi_1(X)$ is the canonical additive character of F_q. A generator of the cyclic group F_q^*, which is the multiplicative group of F_q, is referred to as a primitive element.For each integer j with $0 \leq j \leq q - 2$ and a fixed primitive element α of F_q, the function
$\psi_j(\alpha^i) = \zeta^{ij}$
provides F_q with a multiplicative character,where $q-1$

$0 \leq i \leq q - 2$ *and* $\zeta_{q-1} = e$ —

$2\pi\sqrt{-1}$

$\sqrt{q-1}$.

F_q^* represents the group made up of all multiplicative characters of F_q. When
$q - 1$
$j = ,$ ———
2
The quadratic character is obtained.
$\eta = \psi q \ \ 1$.
2 =
To expand this quadratic character, we set
$\eta(0) = 0$.
The Gauss sum over F_q for all $\psi \in \hat{F}_q$ is provided by

$$g(\psi) = \sum_{X \in F_q^*} \psi(x)\chi_1(X).$$

Griesmer Bound[3-7,12-19]
Linear codes over F_q satisfy the Griesmer bound. A code is called optimum if it attains this bound with equality, while it is distance-optimal if no code with the same length and dimension but a larger minimum distance exists. A general construction of linear codes over finite rings is based on defining sets. For a finite ring R, the code is defined as
$C_K = \{(T_R(ak_1), T_R(ak_2), \ldots, T_R(ak_n)) : a \in R\}$ (1)
, where K is the defining set and $TR(\cdot)$ is the trace function. The resulting code forms an R^n-module, and when it is an ideal of the group ring $R[G]$ for a finite abelian group G, it is called an abelian code. In this construction, the alphabet ring is $R = F_q + uF_q$ with $u^2 = 0$, where $r = q^m$, $R = F_r + uF_r$, and $s = q^k$ with $k \mid m$. The trace function from F_s to F_q is denoted by $Tr_{s/q}$. Four defining sets are considered for constructing the codes:
(i) $K = D + uF_r$, where $D = X \in F_s : Tr_{s/q}(X) = c$ for all $c \in F_q$.
(ii)$K = F_s \ F_q + uF_r$.
(iii)$K = SQF_s \setminus F_q + uF_r$, where $SQ(F_s \setminus F_q)$ denotes the set of all $F_s \setminus F_q$.
(iv)$K = C^{(N,s)} + uF$, where $C^{(N,s)}$ is a cyclotomic coset of F .
$0 \quad r \ 0 \qquad\qquad s$
A cyclotomic coset of F_s. These defining sets generate different families of linear codes with useful algebraic structures and good error-correcting capabilities.

Singleton Bound [8]
Let C be a linear $[n; k; d]$-code over $GF(q)$, with d being its lowest Hamming weight.Therefore, $w(x) \leq n - k + 1$. This paper's primary goal is to determine a comparable bound for a general weight function $W(x)$ over R's minimum weight.

Remark [6]

(i) Let $D = \{X \in F_S^* : Tr_{S/q}(X) = 0\}$. Then the set C_K defined by (1) is a

linear code of length $r \ \frac{s}{q} - 1$ over $F_q + uF_q$. The Lee weight distribution is as

a^2

follows: $W_L(c_a) = 0$ if $a = 0$ or $a = ua_2$ with $Tr_{r/s}(a_2) \in F_q$; $W_L(c_a) = \frac{2r(q-1)s}{q}$ if $a = ua_2$ and $Tr_{r/s}(a_2) \in F_s \setminus F_q$; and $W_L(c_a) = \frac{2r(q-1)(s-q)}{q}$ for $a \in R^*$.
Applying the Gray map $\varphi : R^n \to F_q^{2n}$ to C_K yields the linear code $\varphi(C_K)$ over F_q. The following theorem presents its parameters and establishes its optimality with respect to the Griesmer and Singleton bounds.

(ii) Let $D = F_s \setminus F_q$. The set C_K defined by (2) is a linear code of length $r(s - q)$ over $F_q + uF_q$. The Lee weight distribution of a codeword c_a is determined as follows: $W_L(c_a) = 0$ when $a = 0$, or when $a = ua_2$ with $a_2 \in F_r^*$ and $\mathrm{Tr}_{r/s}(a_2) = 0$. If $a = ua_2$, where $a_2 \in F_r^*$, $\mathrm{Tr}_{r/s}(a_2) \neq 0$, and $\mathrm{Tr}_{r/q}(a_2) = 0$, then

$$W_L(c_a) = \frac{2(q-1)rs}{q}.$$

For $a = ua_2$ with $\mathrm{Tr}_{r/s}(a_2) \neq 0$ and $\mathrm{Tr}_{r/q}(a_2) \neq 0$, as well as for $a \in R^*$, the Lee weight is

$$W_L(c_a) = \frac{2(q-1)(s-q)r}{q}.$$

Thus, the code has a small number of distinct Lee weights, which helps in studying its Gray image and optimality properties.

(iii) Let $K > 1$ and q be odd. Then $D = SQ_{FS/Fq}$. The set C_k described by (2) is a linear code of length $\frac{r(s-q)}{2}$ over $F_q + uF_q$. The Lee weight distribution of a codeword c_a of C_k is determined as follows: $W_L(c_a) = 0$ if $a = 0$, or if $a = ua_2$ with $a_2 \in F_r^*$ and $Tr_{r/s}(a_2) = 0$. If $a = ua_2$, $a_2 \in F_r^*$, $Tr_{r/s}(a_2) \neq 0$, and $Tr_{r/q}(a_2) = 0$, then

$$W_L(c_a) = \frac{(q-1)rs}{q}.$$

For $a = ua_2$ with $Tr_{r/s}(a_2) \neq 0$ and $Tr_{r/q}(a_2) \neq 0$, as well as for $a \in R^*$, the Lee weight is

$$W_L(c_a) = \frac{(q-1)(s-q)r}{q}.$$

Therefore, C_k is a few-weight linear code over $F_q + uF_q$, and its Gray image provides a basis for constructing optimal linear codes.

(iv) Let $N \mid (q-1)$, $\gcd(N, k) = 1$, and $D = C^{(N,s)}$. The set C_k defined by (2) is a linear code of length $\frac{r(s-1)}{0}$ over $F_q + uF_q$. The Lee weight distribution of a codeword C_a is determined as follows: $W_L(C_a) = 0$ when $a = 0$, or when $a = ua_2$ with $a_2 \in F_r^*$ and $\mathrm{Tr}_{r/s}(a_2) = 0$. If $a = ua_2$ with $a_2 \in F_r^*$ and $\mathrm{Tr}_{r/s}(a_2) \neq 0$, then

$$W_L(C_a) = \frac{2(q-1)rs}{q^N}.$$

For $a \in R^*$, the Lee weight is

$$W_L(C_a) = \frac{2(q-1)(s-1)r}{q^N}.$$

Thus, C_k is a few-weight linear code over $F_q + uF_q$, and its Gray image can be used to construct linear codes with desirable parameters.

3. Four Classes of Linear Codes

The distributions of Lee weights for four types of linear codes over R. Four classes of optimum linear codes over F_q may be found using the Gray map. Let $Tr(\cdot)$ be the trace function from R to R. C_K is the linear code over is defined by where

$$R = F_r + uF_q$$

$$C_K = \{ c_a = (Tr(ak))_{k \in K} : a \in R \}. \qquad (2)$$

$$R = F_r + uF_r, \qquad K = D + uF_r.$$

The rank of C_K is equal to that of the R-module encompassed by K, according to the same reasoning as in Theorem 3.3 in [4]. Before determining the Lee weight distribution of C_K,

I: $D = \{ X \in F^* : Tr_{s/q}(X) = c \}$, for any $c \in F_q$.
s

Theorem 3.1
Consider the same hypotheses as in Remark(i). Consequently, φ(C_K) is a

$$\frac{2r(s-1)}{q}, m+k-1,$$

$$2r(q-1)(s-q)q^2$$

ideal linear code on Fq. This means that if we substitute a prime cube for q, we obtain, it is a set φ(CK)

$$\frac{2r(s-1)}{2^3}, m+k-1, 2r(2^3-1)(s-2^3)$$

optimal linear code over Fq with the weight distribution in Table 1.

Proof. We begin to consider the epimorphism

$$\tau : R \to CK, \quad a \mapsto ca.$$

The Fundamental Theorem of Homomorphism for Modules states that

$$R / \ker(\tau) \sim= CK,$$

where

$$\ker(\tau) = \{a \in R : \tau(a) = 0\}.$$

The goal of the proof is to demonstrate that φ(CK) is an ideal code. We assert that

$$\sum_{j=0}^{m+k-2} \left\lceil \frac{2r(2^3-1)(s-2^3)}{(2^3)^2} \right\rceil + 1 > \frac{2r(s-1)}{2^3} - 1.$$

The Griesmer bound. Actually, we have

$$\sum_{j=0}^{m+k-2} \left\lceil \frac{2r(2^3-1)(s-2^3)}{(2^3)^2} \right\rceil = \sum_{j=0}^{m+k-2} \frac{2(2^3-1)}{(2^3)^{m+k-2-j} - (2^3)^{m-1-j}} + 1$$

$$= \frac{2r(s-1)}{2^3} - 1 + m - 1 > \frac{2r(s-1)}{2^3} - 1.$$

Weight W	**Multiplicity** A_W
0	1

$\frac{7rs}{2^3 \cdot 4}$	$\frac{s - 2^3}{2^3}$
$\frac{7r(s - 2^3)}{2^3 \cdot 4}$	$\frac{s(r - 1)}{2^3}$

Table 1: The weight distribution of the code

Theorem 3.2
Using the same hypothesis as in Remark (i). Consequently, φ(CK) is a

$$\frac{2r(s-1)}{2^3}, m + k - 1, 2r(2^3 - 1)(s - 2^3)(2^3)^2$$

optimal linear code over Fq with the weight distribution displayed in Table 2.

Proof. The Standard Singleton bound for every linear code says that $d \leq n - k + 1$.

The parameters to view the explicit relationship,

$$\frac{2r(2^3 - 1)(s - 2^3)}{(2^3)^2} \leq$$

$$\frac{32r(s - 2^3)}{2^3} - (m + k - 1) + 1,$$

$$\frac{2r(s - 2^3)}{2} \leq 2^3 - (m + k - 1) + 1.$$

Example 3.1
Let $q = 2^3$, $m = 2$, and $k = 2$. Then, the comparison of the two bounds is made for the [448,2,392] linear code over$F_2$3 with weight enumerator $1+56Z^{392} + 7Z^{448}$.
Thus, the Griesmer bound is more robust than the Singleton bound.

Weight W	**Multiplicity** A_W
0	1
$\frac{2r(r - 1)(s - 2^3)}{(2^3)^2}$	$(2^3)^k(r - 7) - r$
$\frac{2r^2(s - 2^3)}{(2^3)^2}$	$(2^3)^k(2^3 - r) + r - 1$

Table 2: The weight distribution of the code

II: $D = F_s \setminus F_q$

Theorem 3.3
According to this work, if q is a prime cube, we obtain the following result. With the same symbols as in Remark (ii), the Gray image φ(CK) is an optimal linear code over F_{2^3} with parameters

$$\frac{2r(s - 2^3), m + k, 2(2^3 - 1)(s - 2^3)r}{2^3}$$

The weight distribution of φ(CK) is presented in Table 3.

Proof. The purpose of the proof is to show that φ(Ck) is an ideal code. We claim that

m+k−1

$$\sum_{j=0} \frac{2r(2^3 - 1)(s - 2^3)}{(2^3)j+1}$$

$$> 2rs - 1.$$
$$2^3$$

By the Griesmer bound construction, we have

$$\sum_{j=0}^{m+k-1} \frac{2r(2^3 - 1)(s - 2^3)}{(2^3)j+1}$$

$$= \sum_{j=0}^{m+k-1} \frac{2(2^3 - 1)(2^3)m+k(s - 2^3)}{(2^3)j+1}$$

$$< 2r(s - q).$$

Theorem 3.4
Using the same assumptions as in Remark (ii), we obtain that φ(Ck) is an

$$\frac{2r(s - 2^3), m + k, 2(2^3 - 1)(s - 2^3)r}{2^3}$$

optimal linear code over F_{2^3} with the weight distribution given in Table 4.

Proof. We begin by considering the epimorphism

$$\tau : R \longrightarrow Lc, \quad a \longrightarrow ca.$$

By the Fundamental Theorem of Homomorphisms for modules, we have

$$R / \ker(\tau) \sim= Ck,$$

where

$$\ker(\tau) = \{a \in R : ca = 0\}.$$

According to Remark (ii), it remains to prove that φ(Ck) is an optimal code. We establish this by verifying that

$$\frac{2r(2^3 - 1)(s - 2^3)}{2^3} \leq 2r(s - 2^3) - (m + k) + 1.$$

Equivalently,

$$2^3 \leq 2r(s - 2^3) - m - k + 1.$$

Weight W	**Multiplicity** A_W
0	1
$\frac{7rs}{4}$	$\frac{s - 2^3}{2^3}$ s
$2r(s - 2^3)$	$8r - \frac{}{2^3}$

Table 3: The weight distribution of the code

Weight W	**Multiplicity** A_W
0	1
$\frac{2r(r-1)(s-2^3)}{2^3}$	$(2^3)^k(r-7)-r$
$\frac{2r^2(s-2^3)}{2^3}$	$(2^3)^k(2^3-r)+r-1$

Table 4: The weight distribution of the code

Example 3.2
Let q = 2³, m = 2, and k = 2. Then, a comparison of the two bounds is made for the [3584, 2, 3136] linear code over F_2 with weight enumerator 1 + 56Z3136 + 7Z3584.

Indeed, for a binary linear code with parameters [n, k, d], the Griesmer bound is given by

$$n \geq \sum_{j=0}^{k-1} \left\lceil \frac{d}{2^j} \right\rceil$$

For the code [3584, 2, 3136], the Griesmer bound gives

$$\sum_{j=0}^{3136} \left\lceil \frac{3136}{2^j} \right\rceil$$

$$= \lceil 3136 \rceil + \lceil 1568 \rceil = 4704.$$

On the other hand, the Singleton bound gives

$$d \leq n - k + 1 = 3584 - 2 + 1 = 3583.$$

Hence, the Griesmer bound provides a stronger restriction than the Singleton bound, demonstrating the superiority of the Griesmer bound in this case.

III: D = SQFs/Fq

Theorem 3.5
Using the notation introduced in Remark (iv), we obtain that φ(Ck) is a

$$\left[r(s-2^3),\ m+k,\ \frac{(2^3-1)(s-2^3)r}{2^3} \right]$$

linear code over F_{2^3}. This code attains equality in the Griesmer bound and hence is an optimal linear code.

Proof. The weight distribution of φ(Ck) can be obtained by applying the same argument as in the proof of Theorem 3.3. We now prove the optimality of φ(Ck).

$$\sum_{j=0}^{m+k-1} \frac{r(2^3-1)(s-2^3)}{(2^3)^{j+1}}$$

$$= \sum_{j=0}^{m+k-1} (2^3-1)(2^3)^{m+k-1-j} - (2^3)^{m-j}$$

$$= r(s-2^3).$$

Weight W	**Multiplicity** A_W
0	1
$\frac{rs(2^3 - 1)}{2^3}$	$\frac{s - 2^3}{2^3}$
$\frac{7r(s - 2^3)}{2^3}$	$sr - \frac{s}{2^3}$

Table 5: The weight distribution of the code

Theorem 3.6
Following the notation established in Remark (iii), we obtain that φ(Ck) is a

$$r(s - 2^3),\ m + k,\ \frac{(2^3 - 1)r(s - 2^3)}{2^3}$$

linear code over F_{2^3}. This code attains equality in the Singleton bound and, therefore, satisfies the Singleton bound.

Proof.

$$\frac{r(2^3 - 1)(s - 2^3)}{2^3} \leq r(s - 2^3) - (m + k) + 1.$$

$$2^3 \leq r(s - 2^3) - (m + k) + 1.$$

Weight W	**Multiplicity** A_W
0	1
$\frac{49r}{2^3}$	$2^3 \ (2^3)^{m+k-1} - 1$
$7r$	$(2^3)^{m+k-1} - 1$

Table 6: The weight distribution of the code

Example 3.3
Let $q = 2^3$, $m = 2$, and $k = 2$. Then, a comparison of the two bounds is made for the [3584, 2, 3136] linear code over F_2 with weight enumerator 1 + 56Z3136 + 7Z3584. Thus, the Griesmer bound is therefore stronger than the Singleton bound.

Indeed, for a binary linear code with parameters [n, k, d], the Griesmer bound is given by

$$\sum_{j=0}^{k-1} \lceil d / 2^j \rceil$$

$$n \geq$$

For the code [3584, 2, 3136], the Griesmer sum is

$$\sum_{j=0}^{3136} \lceil 3136 / 2^j \rceil$$

$$= \lceil 3136 \rceil + \frac{3136}{2}$$

$$= 3136 + 1568 = 4704.$$

The Singleton bound gives

$$d \leq n - k + 1 = 3584 - 2 + 1 = 3583.$$

Since the Griesmer bound gives a stronger restriction on the parameters than the Singleton bound in this case, the code satisfies the Griesmer bound and is therefore optimal with respect to this bound.

IV: $D = C^{(N,s)}$.

Let N be a divisor of $s^0 - 1$ and let θ be a primitive element of F_S^*. Define $C^{(N,q)} = \theta^i\langle\theta^N\rangle$, $0 \le i \le N - 1$, where $\langle\theta^N\rangle$ is a cyclic group generated by θ^N.

Theorem 3.7
Using the notation introduced in Remark (iv), we obtain that φ(Ck) is an

$$\frac{2r(s-1),\ m+k,\ N}{2(2^3-1)(s-1)r}$$

$$(2^3)N$$

optimal linear code over F_{2^3} with the weight distribution given in Table 7. Moreover, when N > 1, the parameters of this code attain equality in the Griesmer bound.

Proof.
It is sufficient to prove that φ(Ck) is optimal in order to determine the weight distribution of φ(Ck). If N > 1, then

$$\sum_{j=0}^{m+k-1} \frac{2(2^3-1)r(s-1)}{(2^3)j+1N}$$

$$= \sum_{j=0}^{m+k-1} \frac{2(2^3-1)}{N}$$

$$= \frac{2r(s-1)}{N(2^3)m+k-1-j - (2^3)m-1-j}.$$

Weight W	**Multiplicity** A_W
0	1
$14rs$ / $(2^3)^N$	$s - 1$
$14r(s - 1)$ / $(2^3)^N$	$sr - s$

Table 7: The weight distribution of the code

Theorem 3.8
With the notation introduced in Remark (iv), we obtain that φ(Ck) is an

$$\frac{2r(s-1)\ ,\ m+k,\ N}{2(2^3-1)r(s-1)}$$

linear code over F_{2^3}. Moreover, when N > 1, the parameters of this code attain equality in the Singleton bound.

Proof.

We have

$$\frac{2r(s-1)}{2(2^3-1)r(s-1)} \leq \frac{2r(2^3-1)(s-1)}{N} - (m+k)+1.$$

$$\frac{(2^3)N-(2^3-1)N}{N} - (m+k)+1$$

we obtain

$$\frac{(2^3)N-(m+k)+1}{N}$$

$$= 2r(s-1)/N(2^3)N-(m+k)+1.$$

Weight W Multiplicity A_W

0 1

98r / (2³)N (2³)N − 7N − 1

3
m+k
112r / N(2³)N (7N + 1) (2) − 1

$$\frac{2r(s-1)}{N} - (m+k)+1$$

Weight W	**Multiplicity** A_W
0	1
$\frac{98r}{(2^3)^N}$	$\frac{(2^3)^N - 7N - 1}{(2^3)^N}(2^3)^{m+k}$ (2) 1
$\frac{112r}{N(2^3)^N}$	$(7N+1)\ (2^3)^{m+k} - 1$

Table 8: The weight distribution of the code

2.1 Example 3.4
Let $q = 2^3$, $m = 2$, and $k = 2$. A comparison between the two bounds is performed for the [4032, 2, 3528] linear code over F_23 with weight enumerator $1 + 56Z^{3528} + 7Z^{4032}$. Consequently, the Griesmer bound is more robust than the Singleton bound.

Conclusion

In conclusion, a comparison of the Griesmer and Singleton bounds is used to explore four classes of two-weight threefold linear codes. The Griesmer bound offers a more useful criterion than the Singleton bound for the codes under con-sideration, according to the obtained results. This analysis emphasizes the built-in linear codes' outstanding parameters and optimality.

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